NAVIER-STOKES AND ECONOMIC GROWTH DYNAMICS

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Abstract— Mathematical models of dynamics employing exterior calculus are mathematical representations of the same unifying principle; namely, the description of a dynamic system with a characteristic differential one-form on an odd-dimensional differentiable manifold leads, by analysis with exterior calculus, to a set of differential equations and a characteristic tangent vector which define transformations of the system. This principle is applied to Economic Growth Dynamics and Navier-Stokes Dynamics.

Keywords— Exterior Calculus, Navier-Stokes, Economic Growth.

I. INTRODUCTION

Mathematical models of dynamics employing exterior calculus are mathematical representations of a unifying principle; namely, the description of a dynamic system with a characteristic differential one-form on an odd-dimensional differentiable manifold leads, by analysis with exterior calculus, to a set of differential equations and a characteristic tangent vector which define transformations of the system [1], [2]. The model is here applied to economic growth dynamics and Navier-Stokes dynamics; the outcomes cannot be obtained without exterior calculus.

II. DYNAMICS ON DIFFERENTIAL ONE-FORMS

2. 1 Differential One-Forms

At each point ofa n-dimensional differentiable manifold M with n local coordinates x^k,
(a) there exists a basis set of tangent vectors $\{\partial/\partial \mathbf{x}^\mathbf{k}\}$ for an n-dimensional vector space of tangent vectors $\mathbf{v}$ belonging to tangent space TM_x, and (b) there exists a basis set of differential one-forms $\{\mathbf{d}x^k\}$for an n-dimensional vector space of differential one-forms $\mathbf{d}f$ on tangent space TM_x.
The tangent bundle TM (= $\cup TM_x$) and cotangent bundle T^*M (= $\cup T^*M_x$), where T^*M_x is the dual of TM_x, have the structure of a differential manifold of dimension $2n$ with local coordinates $\{x^k, \mathbf{d}x^k(\mathbf{v})\}$ and $\{x^k, \mathbf{d}f\ (\partial/\partial \mathbf{x}^\mathbf{k})\}$, respectively. Differential one-form $\mathbf{d}S$ on $T^*M_x$is defined by the contraction $\mathbf{d}S\ (\xi)= \mathbf{d}f\ (\mathbf{v})$ where$\xi \in T\ (T^*M_x)$; hence,

$$\mathbf{d}S = \mathbf{d}f\,(\partial/\partial x^k)\,\mathbf{d}x^k \tag{2.1}$$

2.2 Dynamics

For dynamic systems, a temporal coordinate x^0 is introduced as an additional local coordinate for M, TM and T^*M, thereby changing TM and T^*M into odd-dimensional manifolds. Hence, $\mathbf{d}f(\partial/\partial x^0)\mathbf{d}x^0$ is added to eqn.(2.1), where $\mathbf{d}f\ (\partial/\partial x^0)$ is a function of all $(2n+1)$ coordinates;$\mathbf{d}f\ (\partial/\partial x^0)$ describes the phase flow on this extendedcotangent bundle. Using $b_k$for $\mathbf{d}f(\partial/\partial \boldsymbol{x}^k)$ and $\Omega\mathbf{d}x^0$for $\mathbf{d}f(\partial/\partial x^\mathbf{0})\mathbf{d}x^0$, $\mathbf{d}S$ becomes

$$\mathbf{d}S = b_k\,\mathbf{d}x^k + \Omega(x^0, \ldots, x^n; b_1, \ldots, b_n)\mathbf{d}x^0 \tag{2.2}$$

Variable$b_k(t)$ (compare to momentum in Hamiltonian mechanics) is conjugate to the “position” variable $x^k(t)$, since:
(a) $b_k = b_k(x^0)$ and $x^k = x^k(x^0)$
(b) $\Omega = \Omega\ (b_k, x^k, x^0)$
(c) $b_k = \mathbf{d}S\ (\partial/\partial \mathbf{x}^\mathbf{k})$ = contraction of $\mathbf{d}S$ with tangent vector $\partial/\partial \boldsymbol{x}^k$.
The exterior derivative of $\mathbf{d}S$ ($\omega \equiv \mathbf{d}S$) gives the differential two-form $\mathbf{d}\omega$.

$$\mathbf{d}\omega = \mathbf{d}b_k \wedge \mathbf{d}x^k + [(\partial\Omega/\partial x^k)\,\mathbf{d}x^k + (\partial\Omega/\partial b_k)\,\mathbf{d}b_k + (\partial\Omega/\partial x^0)\,\mathbf{d}x^0] \wedge \mathbf{d}x^0 \tag{2.3}$$

where $\omega \equiv \mathbf{d}S$. If x^k and b_k describe mappings of the temporal coordinate onto the direction of the system phase flow, then (a) $x^k = x^k(x^0)$, $b_k = b_k(x^0)$ and (b) the following contraction must hold at each point (b_k, x^k, x^0) of the transformation:

$$\mathbf{d}\omega\ (\xi, \eta) = 0 \tag{2.4}$$

where $$\xi = (\mathrm{d}b_k/\mathrm{d}x^0)\partial/\partial \boldsymbol{b}_k + (\mathrm{d}x^k/\mathrm{d}x^0)\partial/\partial \boldsymbol{x}^k + \partial/\partial \boldsymbol{x}^\mathbf{0} \tag{2.5}$$

and η is an arbitrary vector. $\mathbf{d}\omega$ is a mapping of a pair of vectors onto an oriented surface; if the contraction $\mathbf{d}\omega\ (\xi, \eta) = 0$, then the mapping is defined *only* if the coordinates $\mathrm{d}b_k/\mathrm{d}x^0$ and $\mathrm{d}x^k/\mathrm{d}x^0$ of ξ have the values

$$\mathrm{d}x^k/\mathrm{d}x^0 = -(\partial\Omega/\partial b_k)\ ;\ \mathrm{d}b_k/\mathrm{d}x^0 = (\partial\Omega/\partial x^k) \tag{2.6}$$

Substituting coordinate values from eqns.(2.6) into (2.5), vortex vector $\mathbf{R}(\xi \rightarrow \mathbf{R})$ is

$$\mathbf{R} = (\partial\Omega/\partial x^k)\,\partial/\partial \boldsymbol{b}_k - (\partial\Omega/\partial b_k)\,\partial/\partial \boldsymbol{x}^k + \partial/\partial \boldsymbol{x}^\mathbf{0} \tag{2.7}$$

From the foregoing discussion note that contraction of $\mathbf{d}S$ with vortex vector ($\mathbf{R}$), gives

$\mathbf{d}S(\mathbf{R}) = -b_k\,(\partial\Omega/\partial b_k) + \Omega$(2.8)

where $\mathbf{d}S(\mathbf{R})$ is the Lagrangian on extended tangent space (x^k, dx^k/dx^0, x^0). Also, note that for eqn. (2.4) (where the exterior derivative of a characteristic differential one-form is contracted on a pair of tangent vectors and set equal to the unique scalar zero), the analysis refers to vortex tubes that do not end. For vortex tubes which end in an elementary volume, $\mathbf{d}\omega$ (ξ, η) is set equal to a unique scalar other than zero. A previous application [2] of the present model to the source dependent Maxwell equations illustrates the difference in procedure required for such vortex tubes.

III. ECONOMIC GROWTH DYNAMICS ON A DIFFERENTIAL ONE-FORM

To construct mathematical models of complex economic systems, some economists employ Hamiltonian mechanics, thermodynamics and statistics [3]; recent thermodynamic developments use differential forms. In the present investigation, exterior calculus and its' main tool (differential forms) are used to construct a mathematical model of economic growth dynamics. Using the growth function $Y(K_i, L^i, t)$, the differential one-form proposed for economic growth dynamics is

$\mathbf{d}S = K_i\mathbf{d}L^i - Y\mathbf{d}t$(3.1)

where S plays the role of the action in Hamiltonian mechanics, $Y(K_i, L^i, t)$ is the growth function (the Omega function, e.g., the Hamiltonian), K_i is the capital, L^i is the labor, and t is the time. Use of the proposed principle implies:

Differential one-form	$\mathbf{d}S = K_i\,\mathbf{d}L^i - Y\,\mathbf{d}t$
Basis vectors in tangent space	$\left(\frac{\partial}{\partial K_i}, \frac{\partial}{\partial L^i}, \frac{\partial}{\partial t}\right)$
Differential equations	$\frac{dL^i}{dt} = \frac{\partial Y}{\partial K_i}$; $L^i = +(\partial Y/\partial K_i)\,t +$ constant; $\frac{dK_i}{dt} = -\frac{\partial Y}{\partial L^i}$; $K_i = -(\partial Y/\partial L^i)\,t +$ constant
Vortex vector	$\mathbf{R} = -\left(\frac{\partial Y}{\partial L^i}\right)\frac{\partial}{\partial K_i} + \left(\frac{\partial Y}{\partial K_i}\right)\frac{\partial}{\partial L^i} + \frac{\partial}{\partial t}$
Lagrangian	$\mathbf{d}S(\mathbf{R}) = K_i(\partial Y/\partial K_i) - Y$

IV. NAVIER-STOKES DYNAMICS ON A DIFFERENTIAL ONE-FORM

4.1 Introduction

In fluid dynamics, the Euler and Navier-Stokes equations (NSE) model the dynamics of a fluid in $\mathbf{R}^n$(n = 2 or 3)for times $t \geq 0$.

For incompressible fluids the NSE are

$$\frac{\partial \mathbf{v}}{\partial t} = -(\mathbf{v}\cdot\boldsymbol{\nabla}) + \left[\boldsymbol{\nabla}P + \nu\sum_j^n \frac{\partial}{\partial x^j}\frac{\partial \mathbf{v}}{\partial x^j} + \mathbf{f}\right] \quad (4.1)$$

$$div\,\mathbf{v} = \mathbf{0} \quad (4.2)$$

$$\mathbf{v}(x^1,\dots,x^n,t_0) = \mathbf{v}^0(x^1,\dots,x^n) \quad (4.3)$$

For the case of zero viscosity ν, these equations are the Euler equations. Eqn.(4.3) is the initial condition at positionx^k and time t, eqn.(4.2) is the divergence-free condition and eqn. (4.1) is the equation describing the dynamics, with externally applied force $\mathbf{f}(x^1,\dots,x^n,t)$, velocity $\mathbf{v}(x^1,\dots x^n,t)$, pressure$P(x^1,\dots,x^n,t)$, and with forces due to pressure gradient ∇P and viscous friction $\nu\sum_j \partial_{x^j}\partial_{x^j}\mathbf{v}$.Many studies focus on finding $\mathbf{v}$satisfying the first three equations or on proving or disproving the global existence, smoothness and breakdown of Navier-Stokes solutions on $\mathbf{R}^3$ or on $\mathbf{R}^3/\mathbf{Z}^3$. A critical analysis of many analytic and numerical solutions to eqn. (4.1), led Fefferman[4] to doubt whether standard methods of solving these equations are adequate.

The present investigation is different, namely, the dynamic Navier-Stokes equation is transformed into a differential one-form on an odd-dimensional differentiable manifold. It is then shown that the use of exterior calculus predicts a set of differential equations and tangent vector characteristic of Hamiltonian geometry [1, 2]. This pair of equations is solved for the position x^k as a function of time and for the conjugate to the position$\mathbf{b}_k$ as a function of time. The solution$\mathbf{b}_k$ is shown to be divergence-free by contracting the differential 3-form corresponding to the divergence of the gradient of the velocity with a triple of tangent vectors, implying constraints on two of the tangent vectors for the system. Analysis of solution $\mathbf{b}_k$ shows it is bounded since it remains finite as| x^k |$\rightarrow\infty$, and is physically reasonable since the square of the gradient of the principal function is bounded.

4.2 Differential One-Form

Multiplying eqn. (4.1) by "$-dt$" gives

$$d\mathbf{S} = \mathbf{B}_j\,dx^k - \boldsymbol{\Omega}dt \quad (4.4)$$

where

$$\mathbf{B}_j = (\partial\mathbf{v}/\partial x^j)\ ;\ \mathbf{B}_j dx^j = (\mathbf{v}\bullet\nabla)\mathbf{v}\ dt \quad (4.4.1)$$

$$\boldsymbol{\Omega} = -\boldsymbol{\nabla}P + \nu\sum_j\left(\frac{\partial \mathbf{B}_j}{\partial x^j}\right) + \mathbf{f}$$

$$d\mathbf{S} = -(\partial\mathbf{v}/\partial t)dt \quad (4.4.2)$$

Where $\mathbf{S}$ is referred to as the principal function. To develop $\boldsymbol{\Omega}$ as a function of ($\mathbf{B}_j$, x^j, t) and further characterize the equation for $d\mathbf{S}(x^j, t)$, the quantity $\partial_{x^j}\mathbf{B}_j$ in $\boldsymbol{\Omega}$ is analyzed in the following manner: first Taylor's expansion of $\mathbf{B}_j$ is taken in the neighborhood of initial position ($\mathbf{B}_j(0)$, x_0^j, t_0), then $\partial_{x^j}\mathbf{B}_j$ is taken, then $\partial_{x^j}\mathbf{B}_j(0)$ from Taylor's expansion of $\mathbf{B}_j$ is substituted into the expression for $\partial_{x^j}\mathbf{B}_j$, $[(\mathbf{B}_j(\mathbf{0}) = \mathbf{B}_j(x_0^j, t_0)]$

$$\partial_{x^j}\mathbf{B}_j = \begin{bmatrix} \mathbf{B}_j - \mathbf{B}_j(\mathbf{0}) - \\ (\boldsymbol{t} - \boldsymbol{t}_0)\boldsymbol{\partial}_t\mathbf{B}_j(\mathbf{0}) \end{bmatrix} \left(\boldsymbol{x}^j - \boldsymbol{x}_0^j\right)^{-\mathbf{1}} + \sum_{N=2}^{\infty}\sum_{r=0}^{N} \frac{\boldsymbol{N}-\boldsymbol{r}-\mathbf{1}}{\boldsymbol{r}!\,(\boldsymbol{N}-\boldsymbol{r})!}\left(\boldsymbol{x}^j - \boldsymbol{x}_0^j\right)^{N-r-1} \mathrm{X}(\boldsymbol{t}-\boldsymbol{t}_0)^r \boldsymbol{\partial}_{x^j}^{N-r}\boldsymbol{\partial}_t^r \mathbf{B}_j(\mathbf{0}) \quad (4.5)$$

Substituting this $\partial_{x^j}\mathbf{B}_j$ into $\boldsymbol{\Omega}$ gives

$$\boldsymbol{\Omega} = -\boldsymbol{\nabla}P + \mathbf{f} + \boldsymbol{\nu}\sum_{j=1}^{n}\begin{bmatrix} \mathbf{B_j} - \mathbf{B_j}(0) \\ -(t-t_0)\partial_t\mathbf{B_j}(0)\end{bmatrix}\left(x^j - x_0^j\right)^{-1} + \boldsymbol{\nu}\sum_{j=1}^{n}\sum_{N=2}^{\infty}\sum_{r=0}^{N}\frac{\boldsymbol{N}-\boldsymbol{r}-\mathbf{1}}{\boldsymbol{r}!\,(\boldsymbol{N}-\boldsymbol{r})!}\left(x^j - x_0^j\right)^{N-r-1} \mathrm{X}\ (t-t_0)^r\partial_{x^j}^{N-r}\partial_t^r\mathbf{B_j}(0) \quad (4.6)$$

The one-form corresponding to $d\mathbf{S}$ is

$$\mathbf{dS} = \mathbf{B}_j dx^j - \boldsymbol{\Omega}dt \quad (4.7)$$

where boldface symbol "**d**" is the exterior derivative operator and **dS** is the exterior derivative of vector field **S**. It is noted that $x^j$and$\mathbf{B}_j$ are a conjugate pair, according to the three conditions in sec.(2.2), and since $\mathbf{B}_j = \mathbf{B}_j(x^j, t) = \mathbf{B}_j(x^j(t), t) = \mathbf{b}_j(t)$, then

$$\mathbf{dS} = \mathbf{b}_j dx^j - \boldsymbol{\Omega}dt \quad (4.8)$$

which is analogous to the expression for the differential one-form for the action in Hamiltonian mechanics. The geometric object **dS**is called a vector-valued differential one-form on extended cotangent space T^*M_x(coordinates ($\mathbf{b}_j$, x^j, t)), with basic differential one-forms $\mathbf{db}_j$, $\mathbf{d}x^j$, $\mathbf{d}t$, and characteristic function $\boldsymbol{\Omega}(\mathbf{b}_j, x^j, t)$. With this development, the Navier-Stokes equation is expressed as a differential form useful for applying exterior calculus to solve this equation.

When the technique for applying the proposed principle is utilized, sets of differential equations and a vortex vector is obtained. These results are summarized below:

$$\frac{dx^j}{dt} = \frac{\partial\Omega}{\partial\mathbf{b}_j};\frac{d\mathbf{b}_j}{dt} = -\frac{\partial\Omega}{\partial x^j} \quad (4.9)$$

Using the definition of $\boldsymbol{\Omega}$, with $\mathbf{b}_j$ replacing $\mathbf{B}_j$the above equations become

$$\frac{dx^k}{dt} = \frac{\nu}{(x^k - x_0^k)}\ , \quad (4.10)$$

$$\frac{\boldsymbol{d}\mathbf{b}}{dt} = \partial_{x^k}(\boldsymbol{\nabla}P) - \partial_{x^k}\mathbf{f} - \nu\partial_{x^k}\sum_{j=1}^{n}\begin{bmatrix}\mathbf{b}_j - \mathbf{B}_j(0) \\ -(t-t_0)\partial_t\mathbf{B}_j(0)\end{bmatrix}\left(x^j - x_0^j\right)^{-1} - \nu\partial_{x^k}\sum_{j=1}^{n}\sum_{N=2}^{\infty}\sum_{r=0}^{N}\left(\frac{N-r-1}{r!\,(N-2)!}\right)\left(x^j - x_0^j\right)^{N-r-1} \mathrm{X}\ (t-t_0)^r\partial_{x^j}^{N-r}\partial_t^r\mathbf{B}_j(0) \quad (4.11)$$

The solution to the equation for dx^k/dt is

$$x^k = x_o^k \pm [2\nu(t-t_0)]^{1/2} \quad (4.12)$$

To change the equation for $d\mathbf{b}_k/dt$ so that a series expansion method can be used for its solution, first P and **f**are approximated by a Taylor's series to second order and ∇P is taken, then partial derivatives $\partial_{x^k}\mathbf{f}$ **and** $\partial_{x^k}\nabla P$ andare taken. When comparing the terms $\partial_{x^k}\partial_{x^{k+1}}\mathbf{f}(\mathbf{0})(x^{k+1} - x_0^{k+1})and\partial_{x^{k+2}}\partial_{x^k}\mathbf{f}(0)(x^{k+2} - x_0^{k+2})$with $\left[\partial_{x^k}^2\mathbf{f}(0)\right](x^k - x_0^k)$, all in $\partial_{x^k}\mathbf{f}$, it is assumed $\partial_{x^k}\partial_{x^{k+1}}\mathbf{f}(\mathbf{0}) \ll \partial_{x^k}^2\mathbf{f}(0)$ and $\partial_{x^{k+2}}\partial_{x^k}\mathbf{f}(\mathbf{0}) \ll \partial_{x^k}^2\mathbf{f}(0)$; these terms are excluded since a coordinate system can be supposed where the terms are zero or exceptionally small. Following the above indicated procedure and noting once again that $\mathbf{b}_j = \mathbf{b}_j(t)$, the differential equation for $d\mathbf{b}_k/dt$ becomes

$$\frac{\boldsymbol{d}\mathbf{b}}{dt} = -\sum_{N=2}^{\infty}\sum_{r=0}^{N}\left[\frac{\nu(N-r-1)^2}{r!\,(N-r)!}\partial_{\boldsymbol{x}^k}^{N-r}\partial_t^r\mathbf{B}_k(0)\right] \mathrm{X}\ (x^k - x_0^k)^{N-r-2}(t-t_0)^r + \left[-\partial_{x^k}\partial_t\mathbf{f}(0)\right](t-t_0) + \left[-\partial_{x^k}^2\mathbf{f}(0)\right](x^k - x_0^k) + \left[\left(\mathbf{e_k}\partial_{x^k}^2 + \mathbf{e}_{k+1}\partial_{x^k}\partial_{x^{k+1}} + \mathbf{e}_{k+2}\partial_{x^{k+2}}\partial_{x^k}\right)P(0) - \partial_{x^k}\mathbf{f}(0)\right] + \left[\nu\left(\mathbf{b}_k - \mathbf{B}_k(0)\right)\right](x^k - x_0^k)^{-2} + \left[-\nu\partial_t\mathbf{B}_k(0)\right](x^k - x_0^k)^{-2}(t-t_0) \quad (4.13)$$

$\mathbf{e}_k$ is a unit vector arising from the use of the gradient. Multiplying $d\mathbf{b_k}/\mathbf{dt}$ by $(x^k - x_o^k)^2$ and using $x^k = x_o^k \pm [2\nu(t-t_0)]^{1/2}$ to remove the$(t-t_0)$ dependence, $d\mathbf{b}_k/dt$ becomes

$$(x^k - x_0^k)^2\frac{\boldsymbol{d}\mathbf{b}}{dt} = \left[\nu\left(\mathbf{b}_k - \mathbf{B}_k(0)\right)\right] - \sum_{N=2}^{\infty}\sum_{r=0}^{N}\left[\frac{(N-r-1)^2}{2(2\nu)^{r-1}r!\,(N-r)!}\partial_{\boldsymbol{x}^k}^{N-r}\partial_t^r\mathbf{B}_k(0)\right] \mathrm{X}(x^k - x_0^k)^{N+r}$$

$$+\left[-(2\nu)^{-1}\partial_{x^k}\partial_t\mathbf{f}(0)\right](x^k-x_0^k)^4$$
$$+\left[-\partial_{x^k}^2\mathbf{f}(0)\right](x^k-x_0^k)^3$$
$$+\left[\begin{array}{c}-\frac{1}{2}\partial_t\mathbf{B}_k(0)-\boldsymbol{\partial}_{x^k}\mathbf{f}(0)\\ +\begin{pmatrix}\mathbf{e}_k\partial_{x^k}^2\\ +\mathbf{e}_{k+1}\partial_{x^k}\partial_{x^{k+1}}\\ +\mathbf{e}_{k+2}\partial_{x^{k+2}}\partial_{x^k}\end{pmatrix}P(0)\end{array}\right](x^k-x_0^k)^2 \quad (4.14$$

The series solution to the foregoing equation begins by using a trial $\mathbf{b}_k$.

$\mathbf{b}_k(t)=\sum_{N=1}^{\infty}C_N\,(t-t_0)^{\frac{N}{2}}e^{[-Na(t-t_0)^{1/2}]}$(4.15)

Using $x^k = x_o^k \pm [2\nu(t-t_0)]^{1/2}$, then

$$\mathbf{b}_k(x^k)=\sum_{N=1}^{\infty}\frac{C_N(x^k-x_0^k)^N}{(2\nu)^{N/2}}e^{\left[-\frac{Na}{(2\nu)^{1/2}}(x^k-x_0^k)\right]} \quad (4.16)$$

where the $\mathbf{C}_N$ and *"a»* are constants. Computation of $\frac{d\mathbf{b}_k}{dt}$ with the use of the equation for $\mathbf{b}_k(t)$, followed by use of $x^k = x_o^k \pm [2\nu(t-t_0)]^{1/2}$ to express $\frac{d\mathbf{b}_k}{dt}$ as a function of $(x^k-x_o^k)$, use of the equation for $\mathbf{b}_k(x^k)$, expanding the exponential function by a Taylor's series to second-order, rearranging and combining terms, and changing $(x^k-x_o^k)^{N+r}$ to $(x^k-x_o^k)^N$, changes $(x^k-x_0^k)^2\frac{d\mathbf{b}_k}{dt}$ into:

$$\nu\mathbf{B}_k(0)+\sum_{i=N,4,3,2}A_i\,(x^k-x_0^k)^i=0, \quad (4.17)$$

where --

$$A_N=\sum_{N=5}^{\infty}\left[\left(\sum_{r=0}^{r\le N/2}\frac{2(N-2r-1)^2}{4(2\nu)^{r-1}(N-2r)!}\right)\partial_{x^k}^{N-2r}\partial_t^r\mathbf{B}_k(0)\right]$$
$$-\alpha^3(N-3)^3C_{N-3}+\alpha^2(N-2)^2(N-1)C_{N-2}$$
$$-2a(N-1)^2C_{N-1}+2(N-1)C_N$$

$$A_4=\left(\frac{-a^3\nu^{-1}}{8}\right)C_1+\left(\frac{3a^2\nu^{-1}}{2}\right)C_2+\left(\frac{-9a\nu^{-1}}{4}\right)C_3+\left(\frac{3\nu^{-1}}{4}\right)C_4+(2\nu)^{-1}\partial_{x^k}\partial_t\mathbf{f}(0)$$
$$+\left(\frac{3}{8}\nu\partial_{x^k}^4+\frac{1}{4}\partial_{x^k}^2\partial_t+\frac{1}{4}(2\nu)^{-1}\partial_t^2\right)\mathbf{B}_k(0)$$

$$A_3=\left(\frac{a^2}{2(2\nu)^{1/2}}\right)C_1-\left(\frac{2a}{(2\nu)^{1/2}}\right)C_2+\left(\frac{1}{(2\nu)^{1/2}}\right)C_3+\frac{2}{3}\nu\partial_{x^k}^3\mathbf{B}_k(0)+0+\boldsymbol{\partial}_{x^k}^2\mathbf{f}(0)$$

$$A_2=-\left(\frac{a}{2}\right)C_1+\left(\frac{1}{2}\right)C_2+\partial_{x^k}\mathbf{f}(0)+\frac{1}{2}\left(\nu\partial_{x^k}^2+\partial_t\right)\mathbf{B}_k(0)-\begin{pmatrix}\mathbf{e}_k\partial_{x^k}^2+\mathbf{e}_{k+1}\partial_{x^k}\partial_{x^{k+1}}\\ +\mathbf{e}_{k+2}\partial_{x^{k+2}}\partial_{x^k}\end{pmatrix}P(0)$$

Equating the A_i for each power of $(x^k-x_0^k)$ to zero, the coefficients are obtained in the form $C_n = C_n(C_1)$, giving the complete solution.

4.3 Analysis of Solution

•**Bounded solution:** Using $\mathbf{b}_k = \mathbf{b}_k(x^k)$, the α-th derivative of $\mathbf{b}_k$ is

$$\frac{d^\alpha\mathbf{b}_k}{dx^{k\alpha}}=\sum_{N=1}^{\infty}\sum_{m=0}^{\alpha}\beta_{Nm\alpha}C_N(x^k-x_0^k)^{N-(\alpha-m)}$$
$$\text{X}\quad exp\left(\frac{-aN}{(2\nu)^{1/2}}(x^k-x_0^k)\right) \quad (4.18)$$
$$\beta_{Nm\alpha}=\frac{1}{(2\nu)^{N/2}}\left(\frac{-aN}{(2\nu)^{1/2}}\right)^m\binom{\alpha}{m}\left(\frac{N!}{[N-(\alpha-m)]!}\right) \quad (4.19)$$

and where N ≥α - *m*. At $t = t_0$, $\frac{d^\alpha\mathbf{b}_k}{dx^k}|_{x_0^k}=0$, since $x^k = x_0^k$ at $t = t_0$. Hence, $lim_{|x^k|\to\infty}|\frac{d^\alpha\mathbf{b}_k}{dx^k}|_{x_0^k}|=\mathbf{0}$ for any α, implying $\mathbf{b}_k$ converges as $|x^k|\to\infty$.

•**Incompressibility:** Eqn.(4.2) is the condition for a velocity vector field **v** to be divergence-free. If ∂_{x^k} is taken on each side of (4.2), it becomes *div* $\mathbf{B}_k = 0$. In differential geometry, the divergence of vector field $\mathbf{B}_k$ on the oriented cotangent space T^*M_x is the density in the expression for the 3-form on T^*M_x.

$$w^3=\left(div\ \mathbf{B}_k\right)\mathbf{db}_k \wedge \mathbf{d}x^k \wedge \mathbf{d}t \quad (4.20)$$

where ω^3 defines the sources in an elementary parallelepiped with edges($\varepsilon\xi_\alpha$, $\boldsymbol{\varepsilon}\xi_\beta$, $\varepsilon\xi_\kappa$) and tangent vectors $\xi\in T(T^*M_x)$, where $\mathbf{db}_j$, $\mathbf{d}x^j$ and $\mathbf{d}t$ are basis differential one-forms for cotangent space T^*M_x at point $(x^1,\ldots, x^n)$ of *M*, where $\mathbf{db}_j\wedge\mathbf{d}x^j\wedge\mathbf{d}t$ is the volume form and **ε** is an arbitrarily small number. In order for *div* $\mathbf{B}_k = 0$, then $\mathbf{db}_j\wedge\mathbf{d}x^j\wedge\mathbf{d}t\ (\xi_\alpha, \xi_\beta, \xi_\kappa) = 0$. For tangent vector ξ_α and arbitrary tangent vectors ξ_β, ξ_κ,

$$\xi_\alpha=\left(\frac{d\mathbf{b}_k}{dt}\right)\partial_{\mathbf{b}_k}+\left(\frac{dx^k}{dt}\right)\partial_{x^k}+\partial_t \quad (4.21)$$
$$\xi_\beta=\beta_{\mathbf{b}_k}\left(\frac{d\mathbf{b}_k}{dt}\right)\partial_{\mathbf{b}_k}+\beta_{x^k}\left(\frac{dx^k}{dt}\right)\partial_{x^k}+\partial_t \quad (4.22)$$
$$\xi_\kappa=\kappa_{\mathbf{b}_k}\left(\frac{d\mathbf{b}_k}{dt}\right)\partial_{\mathbf{b}_k}+\kappa_{x^k}\left(\frac{dx^k}{dt}\right)\partial_{x^k}+\partial_t \quad (4.23)$$

then $\mathbf{db}_j\wedge\mathbf{d}x^j\wedge\mathbf{d}t\ (\xi_\alpha, \xi_\beta, \xi_\kappa) = 0$ only if

$$\beta_{\mathbf{b}_k}(\kappa_{x^k}-1)-\beta_{x^k}(\kappa_{\mathbf{b}_k}-1)+(\kappa_{\mathbf{b}_k}-\kappa_{x^k})=0 \quad (4.24)$$

where $(\partial_{\mathbf{b}_k}, \partial_{x^k}, \partial_t)$ are basis tangent vectors for tangent space $T(T^*M_x)$. The condition on $\beta_{x^k}, \beta_{\mathbf{b_k}}, \kappa_{x^k}$, and $\kappa_{\mathbf{b}_k}$ implies that the vectors ξ_3 and ξ_κ are not entirely arbitrary; the condition distorts the parallelepiped $(\xi_\alpha, \xi_\beta, \xi_\kappa)$ to allow the gradient of $\mathbf{v}$ to be divergence-free. In space (x^k, t), the condition on β_{x^k} and κ_{x^k} is

$$\beta_{x(1)}(\kappa_{x(2)} - \kappa_{x(3)}) + \beta_{x(2)}(\kappa_{x(3)} - \kappa_{x(1)}) + \beta_{x(3)}(\kappa_{x(1)} - \kappa_{x(2)}) = 0 \quad (4.25)$$

If the volume of this parallelepiped is in the same region of space in which the motion of the system occurs, then the divergence equation are fulfilled.

(5.19)

The **vortex vector R** and **Lagangian dS(R)** are

$$\mathbf{R} = -(\partial_{x^k}\boldsymbol{\Omega})\partial_{\mathbf{b}_k} + (\partial_{\mathbf{b}_k}\boldsymbol{\Omega})\partial_{x^k} + \partial_t = -(\partial_{x^k}\boldsymbol{\Omega})\partial_{\mathbf{b}_k} + \nu(x^k - x_0^k)^{-1}\partial_{x^k} + \partial_t \quad (4.26)$$

$$\mathbf{dS}(\mathbf{R}) = \mathbf{b}_k\,\partial_{\mathbf{b}_k}\boldsymbol{\Omega} - \boldsymbol{\Omega} = \mathbf{b}_k\nu(x^k - x_0^k)^{-1} - \boldsymbol{\Omega} \quad (4.27)$$

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