SOLVING LINEAR AND NON-LINEAR STIFF SYSTEM OF ORDINARY DIFFERENTIAL EQUATIONS BY MULTI STAGE HOMOTOPY PERTURBATION METHOD

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Abstract- In this paper, linear and non-linear stiff systems of ordinary differential equations are solved by the multi-stage homotopy perturbation method (MHPM). The MHPM is a technique adapted from the standard homotopy perturbation method (HPM) where standard HPM is converted into a hybrid numeric-analytic method called multistage homotopy perturbation method (HPM). The MHPM is tested for several examples. Comparisons with an explicit Runge-Kutta-type method (RK) demonstrate the promising capability of the MHPM for solving linear and non-linear stiff systems of ordinary differential equations.

Index Terms- Stiff system of ODEs, Runge-Kutta-type method, homotopy perturbation method, Multistage HPM.

I. INTRODUCTION

The mathematical equations modelling many real-world physical phenomena are often stiff equations, i.e. equations with a wide range of temporal scales. The numerical methods for solving stiff equations must have good accuracy and wide region of stability. Hojjati et al. [1] developed a multistep method for solving stiff systems of initial value problems (IVPs). Knowing that the classical explicit fourth-order Runge-Kutta method is insufficient for the solution of stiff IVPs, Ahmad et al. [2] presented an explicit Taylor-like method for solving stiff IVPs. In Ahmad and Yaacob [3], an explicit Runge-Kutta-like method is developed and shown to be efficient for the solution of stiff ODEs. Very recently, Nie et al. [4] presented a class of efficient semi-implicit schemes for stiff reaction-diffusion equations. A variable-step size algorithm for stiff systems has been proposed recently by Jannelli and Fazio [5]. In [6], a class of methods having properties very close to those of traditional Runge-Kutta methods were developed. Butcher and Hojjati [7] devised a class of second derivative methods possessing Runge-Kutta stability property. Hojjati et al. [8] presented a new class of second derivative multistep methods with improved stability region.
All of the methods mentioned above need some sort of discretizations. One of the papers proposing an approximate analytic method is due to Guzel and Bayram [9] who presented a power series method for stiff systems. Homotopy perturbation method [10, 11] is an analytical method which can be applied to the solution of linear, nonlinear deterministic and stochastic operator equations. HPM deforms a difficult problem into an infinite set of problems which are easier to solve without any need to transform nonlinear terms. The applications of HPM in nonlinear problems have been demonstrated by many researchers.In recent years, much attention has been devoted to the application of the HPM, to the solutions of various scientific models [13, 14, 15, 16]. HPM yields rapidly convergent series solutions [17]. Very recently, Chowdhury et al. [18], Chowdhury and Hashim [19],Hashim and Chowdhury [20] were first to successfully apply the multistage homotopy-perturbation method (MHPM) to the chaotic Lorenz system, Chen system and a class of systems of ODEs. Very recently in [21], Olvera and Elas-Ziga have introduced the enhanced multistage homotopy perturbation method (EMHPM) approach to solve nonlinear dynamical systems. The mathematical equations modelling many real-world physical phenomena are often stiff equations, i.e. equations with a wide range of temporal scales.

To the best of our knowledge, the standard HPM does not work accurately in solving chaotic, hyper-chaotic systems and stiff systems for long time spans [22, 23]. In this paper, we extend our previous work [23] to apply MHPM for solving linear and non-linear stiff systems of ordinary differential equations. Comparisons will be made against an explicit Runge-Kutta method and available exact solutions.

II. SOLUTION METHODS

In this section we describe the MHPM and give the algorithm of the Runge-Kutta-like method of Ahmad and Yaacob [3] for solving the following initial value problem:

$$y' = f(t, y) \text{ with } y(0) = y_0, \qquad (1)$$

where $f(t,y)$ may be a linear or non-linear function.

A. MHPM for Systems of ODEs

To illustrate the basic concept of the MHPM, consider the general system of first-order ODEs.

$$\begin{aligned} &\frac{du_1}{dt} + g_1(t, u_1, u_2, \cdots, u_m) = f_1(t), \\ &\frac{du_2}{dt} + g_2(t, u_1, u_2, \cdots, u_m) = f_2(t), \\ &\vdots \\ &\frac{du_m}{dt} + g_1(t, u_1, u_2, \cdots, u_m) - f_m(t), \end{aligned} \quad (2)$$

subject to the initial conditions

$u_1(t^*) = c_1, u_2(t^*) = c_2, \cdots, u_m(t^*) = c_m$, (3)

where t^*, is the left-end point of each subinterval.
We construct a homotopy for system (2) in the form

$$\begin{aligned} &L(u_1) - L(v_1) + pL(v_1) + p[N_1(u_1, u_2, \cdots, u_m) - f_1] = 0, \\ &L(u_2) - L(v_2) + pL(v_2) + p[N_2(u_1, u_2, \cdots, u_m) - f_2] = 0, \\ &\vdots \\ &L(u_m) - L(v_m) + pL(v_m) + p[N_m(u_1, u_2, \cdots, u_m) - f_m] = 0, \end{aligned} \quad (4)$$

where $p \in [0, 1]$ is an embedding parameter and $v_1, v_2, \cdots, v_m$ are initial approximations which satisfying the given conditions. It is obvious that when the homotopy parameter $p = 0$, Eqs. (4) become a linear system of equations and when $p = 1$ we get the original nonlinear system of equations.

Consider the initial approximations as follows:

$$\begin{aligned} &u_{1,0}(t) = v_1(t) = u_1(t^*) = c_1, \\ &u_{2,0}(t) = v_2(t) = u_2(t^*) = c_2, \\ &\vdots \\ &u_{m,0}(t) = v_m(t) = u_m(t^*) = c_m \end{aligned} \quad (5)$$

and

$$\begin{aligned} &u_1(t) = u_{1,0}(t) + pu_{1,1}(t) + p^2 u_{1,2}(t) + p^3 u_{1,3}(t) + \cdots, \\ &u_2(t) = u_{2,0}(t) + pu_{2,1}(t) + p^2 u_{2,2}(t) + p^3 u_{2,3}(t) + \cdots, \\ &\vdots \\ &u_m(t) = u_{m,0}(t) + pu_{m,1}(t) + p^2 u_{m,2}(t) + p^3 u_{m,3}(t) + \cdots, \end{aligned} \quad (6)$$

where $u_{i,j} (i = 1, 2, \cdots, m;\ j = 1, 2, \cdots)$ are functions yet to be determined. Substituting (5)–(6) into (4) and arranging the coefficients of the same powers of p, we get

$$\begin{aligned} &L(u_{1,1}) + L(v_1) + N_1(u_{1,0}, u_{2,0}, \cdots, u_{m,0}) - f_1 = 0, \quad u_{1,1}(t_0) = 0, \\ &L(u_{2,1}) + L(v_2) + N_2(u_{1,0}, u_{2,0}, \cdots, u_{m,0}) - f_2 = 0, \quad u_{2,1}(t_0) = 0 \\ &\vdots \\ &L(u_{m,1}) + L(v_m) + N_m(u_{1,0}, u_{2,0}, \cdots, u_{m,0}) - f_{m1} = 0, \quad u_{m,1}(t_0) = 0, \quad (7) \\ &L(u_{1,2}) + N_1(u_{1,1}, u_{2,1}, \cdots, u_{m,1}) = 0, \quad u_{1,2}(t_0) = 0, \\ &L(u_{2,2}) + N_2(u_{1,1}, u_{2,1}, \cdots, u_{m,1}) = 0, \quad u_{2,2}(t_0) = 0, \\ &\vdots \\ &L(u_{m,2}) + N_m(u_{1,1}, u_{2,1}, \cdots, u_{m,1}) = 0, \quad u_{m,2}(t_0) = 0, \end{aligned}$$

etc. Solve the ab ove system of equations for the unknowns $u_{i,j} (i = 1, 2, \cdots, m;\ j = 1, 2, \cdots)$ applying the inverse operator

$$L^{-1}(\cdot) = \int_{t^*}^{t} (\cdot) dt \quad (8)$$

Therefore, the n-term approximations for the solutions of (7) can be expressed as

$$\begin{aligned} &\phi_{1,n}(t) = u_1(t) = \lim_{p \to 1} u_1(t) = \sum_{k=0}^{n-1} u_{1,k}(t), \\ &\phi_{2,n}(t) = u_2(t) = \lim_{p \to 1} u_2(t) = \sum_{k=0}^{n-1} u_{2,k}(t), \\ &\vdots \\ &\phi_{m,n}(t) = u_m(t) = \lim_{p \to 1} u_m(t) = \sum_{k=0}^{n-1} u_{m,k}(t) \end{aligned} \quad (9)$$

In order to carry out the iterations in every subinterval of equal length Δt, $[0,t_1), [t_1,t_2), [t_2,t_3), \cdots [t_{j-1},t)$, we need to know the values of the following,

$$u_{1,0}(t) = u_1(t^*), \ u_{2,0}(t) = u_2(t^*), \cdots, u_{m,0}(t) = u_m(t^*). \quad (10)$$

In general, we do not have this information at our clearance except at the initial point $t^* = t_0 = 0$. A simple way for obtaining the necessary values could be by means of the previous n-term approximations $\phi_{1,n}, \phi_{2,n}, \cdots, \phi_{m,n}$ of the preceding subinterval, i.e.

$$u_{1,0} \cong \phi_{1,n}(t^*), \quad u_{2,0} \cong \phi_{2,n}(t^*), \cdots, \quad u_{m,0} \cong \phi_{m,n}(t^*). \quad (11)$$

B. Runge-Kutta-like method

Ahmad and Yaacob [3] developed composite arithmetic-harmonic, explicit Runge-Kutta-like methods for solving problem (1). The iterative formula they proposed is given as follows.

$$y_{n+1} = y_n + \frac{h}{2}\left(\frac{2k_1k_2}{k_1 + k_2} + \frac{2k_2k_3}{k_2 + k_3}\right), \quad (12)$$

$$k_1 = f(t_n, y_n), \quad (13)$$

$$k_2 = f\left(t_n + \frac{3}{5}h, \ y_n + \frac{3}{5}hk_1\right), \quad (14)$$

$$k_3 = f\left(t_n + \frac{4}{5}h, \ y_n + \frac{4}{5}h\left[\frac{k_1 + k_2}{2}\right]\right), \quad (15)$$

where h is the time stepsize. For further details, the reader is referred to [3].

III. TEST PROBLEMS

In this section, we shall demonstrate how well the MHPM compares with the RungeKutta-like method of [3] for the solutions of both linear and non-linear system of ordinary differential equations (ODES). The MHPM iterative algorithm is coded in the computer algebra package Maple. The Maple environment variable Digits controlling the number of significant digits is set to 16 in all the calculations done in this paper.

A. Problem 1

Now consider linear stiff initial value problem [24]:

$$\frac{dy_1}{dt} = -2000y_1 + 1000y_2 + 1, \quad (16)$$

$$\frac{dy_2}{dt} = y_1 - y_2, \quad (17)$$

subject to the initial conditions

$$y_1(0) = 0, \qquad y_2(0) = 0, \quad (18)$$

The exact solutions of the system (16)-(17) are given by

$$y_1(t) = -4.97\times10^{-4}e^{-2000.5t} - 5.034\times10^{-4}e^{-0.5t} + 0.001, \quad (19)$$

$$y_2(t) = 2.5\times10^{-7}e^{-2000.5t} - 1.007\times10^{-3}e^{-0.5t} + 0.001 \quad (20)$$

The iterative procedure given in section 3 for the system (16)–(17) are given by

$$y_{1,0} = L^{-1}(1), \qquad y_{2,0} = 0, \quad (21)$$

$$y_{1,k+1} = -2000L^{-1}(y_{1,k}) + 1000L^{-1}(y_{2,k}), \quad k \ge 0, \quad (22)$$

$$y_{2,k+1} = L^{-1}(y_{1,k}) - L^{-1}(y_{2,k}), \quad k \ge 0, \quad (23)$$

The solutions based on only 4-term MHPM for the time steps $h = 10^{-3}$ is tabulated in Table 1. The MHPM solutions at $h = 10^{-3}$ are of comparable accuracy with that of the Runge-Kutta method at the same step size. We can improve the accuracy of the MHPM solutions by increasing the number of terms in the series solution.

Table 1: Decomposition solutions using 4 terms as compared with the exact solutions and solutions from the explicit Runge-Kutta method.

t	Exact Solution		MHPM solutions		RK solutions	
	y_1	y_2	y_1	y_2	y_1	y_2
0.1	5.211E-04	4.211E-05	5.242E-04	4.854E-05	5.241E-04	4.852E-05
0.5	6.079E-04	2.157E-04	6.105E-04	2.210E-04	6.103E-04	2.209E-04
1	6.946E-04	3.892E-04	6.967E-04	3.934E-04	6.965E-04	3.932E-04
2	8.148E-04	6.295E-04	8.162E-04	6.322E-04	8.159E-04	6.319E-04
3	8.876E-04	7.753E-04	8.886E-04	7.771E-04	8.883E-04	7.767E-04
4	9.318E-04	8.637E-04	9.326E-04	8.649E-04	9.322E-04	8.645E-04
5	9.586E-04	9.173E-04	9.592E-04	9.183E-04	9.589E-04	9.178E-04

B. Problem 2

Now consider nonlinear stiff initial value problem [25]:

$$\frac{dy_1}{dt} = -12y_1 + 10y_2^2, \quad (24)$$

$$\frac{dy_2}{dt} = y_1 - y_2 - y_1y_2, \quad (25)$$

subject to the initial conditions

$$y_1(0) = 0, \qquad y_2(0) = 0. \quad (26)$$

The exact solutions of the system (24)-(25) are given by

$$y_1(t) = e^{-2t}, \quad (27)$$

$$y_2(t) = e^{-t}. \quad (28)$$

The iterative procedure given in section 3 for the system (24)–(25) are given by

$$y_{1,0} = 0, \quad y_{2,0} = 0, \quad (29)$$

$$y_{1,k+1} = -12L^{-1}(y_{1,k}) + 10L^{-1}(M_k), \quad k \ge 0, \quad (30)$$

$$y_{2,k+1} = L^{-1}(y_{1,k}) - L^{-1}(N_k), \quad k \ge 0, \quad (31)$$

where some of the homotopy polynomials for M_k and N_k are obtained as:

$$M_0 = y_{1,0}^2, \quad M_1 = 2y_{1,0}y_{2,0}, \quad M_2 = 2y_{1,0}y_{1,2} + y_{11}^2,$$

etc.

$$N_0 = y_{1,0}y_{2,0}, \quad N_1 = y_{1,0}y_{2,1} + y_{1,1}y_{2,0}, \quad N_2 = y_{1,0}y_{2,2} + y_{1,1}y_{2,1} + y_{1,2}y_{2,0},$$

etc.

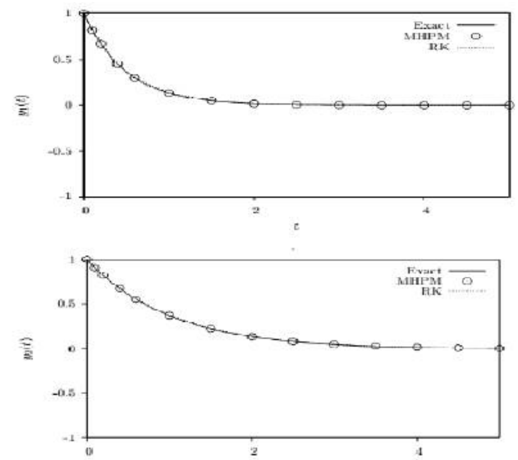

Fig. 1: The MHPM solutions using 4 terms as compared with the exact solutions and that obtained by the Runge-Kutta method for Problem 2.

In Fig. 1 we compare the 4-term decomposition solutions with the exact solutions and that obtained by the Runge-Kutta method. The MHPM solutions at $h = 10^{-3}$ are of comparable accuracy with the exact solutions and that of the Runge-Kutta method at the same step size.

CONCLUSION

In this paper, we presented the multi-stage homotopy perturbation method (MHPM) for solving both linear and non-linear stiff system of ODEs. Direct applications of the classical HPM can fail for stiff problems. The MHPM is shown here to be a promising alternative method for stiff equations. In addition to the choice of time stepsize, the MHPM has the number of terms of the series solution as an extra parameter for controlling the accuracy of solutions.

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